

Adjoint Transfer and Order-Lifting Dichotomies for 2×2 Matrices over Prime-Power Rings

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Abstract

We study determinant-compatible lifts of invertible 2×2 matrices from modulus p^k to modulus p^{k+1} , with p an odd prime. The first variation of the order defect is governed by an affine translate of an adjoint transfer operator on $\mathfrak{sl}_2(\mathbb{F}_p)$. For a non-scalar reduction q , the image of the transfer lies on the traceless centralizer axis. We isolate a discriminant dichotomy. In the regular semisimple case the transfer has rank one. In the non-semisimple Jordan case it vanishes for $p > 3$, whereas characteristic 3 is exceptional: the quadratic moment of the unipotent adjoint action survives and the transfer again has rank one.

For $p = 3$ this yields a uniform order-lifting law for every non-scalar coarse state. The apparent persistence problem for the affine zero fiber can be closed algebraically: the defect of every determinant-compatible reference lift commutes with the coarse reduction and is traceless, hence lies on the traceless centralizer axis; in characteristic three this axis is exactly the image of the full-order adjoint transfer. Therefore the affine zero equation is solvable at every PRE level. Its solution set is a coset of a two-dimensional kernel, so exactly 9 of the 27 lifts preserve the coarse order and 18 triple it. After the first order tripling, the activated state is absorbing. Consequently the recursive law

$$\text{PRE} \longrightarrow 9\text{PRE} + 18\text{ACT}, \quad \text{ACT} \longrightarrow 27\text{ACT}$$

holds unconditionally along every non-scalar characteristic-three lifting tower considered here. The argument separates standard ingredients—congruence tangent spaces, cyclic transfer operators, and prime-power binomial congruences—from the specific affine order-defect mechanism that connects them.

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1 Introduction

Let p be an odd prime. A basic lifting problem asks how the order of a matrix modulo p^k changes when the matrix is lifted to modulus p^{k+1} . The purpose of this paper is to organize this question through a tangent-space description of determinant-compatible lifts.

The relevant tangent space is $\mathfrak{sl}_2(\mathbb{F}_p)$. The first-order change of the order defect is not a bare linear invariant: it is an *affine* translate of a cyclic adjoint transfer. This distinction is essential for obtaining exact fiber counts.

The central phenomenon is especially rigid in characteristic 3. For every non-scalar reduction the relevant transfer has rank one. Consequently a three-dimensional tangent space splits into affine fibers of size $3^2 = 9$, producing the exact count

$$27 = 9 + 18.$$

After the first order increase, a periodic-transfer cancellation and a prime-power propagation identity make the activated state absorbing.

The standard background on congruence filtrations and modular cyclic representations is cited below. The novelty claim of the present draft is deliberately conservative: the focus is the specific connection between affine order defects, adjoint transfer, discriminant type, and exact order-lifting branching. A final priority claim should be made only after a complete literature audit.

2 Setup

Fix an odd prime p . Let

$$G_{p^k} = \{L \in GL_2(\mathbb{Z}/p^k\mathbb{Z}) : \det L \in \{+1, -1\}\}.$$

We consider compatible lifts from G_{p^k} to $G_{p^{k+1}}$.

Let $q \in GL_2(\mathbb{F}_p)$ be non-scalar. Define

$$A_q = q - \frac{\text{tr } q}{2}I, \quad \Delta(q) = (\text{tr } q)^2 - 4 \det q.$$

Cayley–Hamilton gives

$$A_q^2 = \frac{\Delta(q)}{4}I. \tag{1}$$

For an integer $r \geq 1$, define the length- r adjoint sum

$$N_q^{(r)}(X) = \sum_{j=0}^{r-1} q^{-j} X q^j, \quad X \in \mathfrak{sl}_2(\mathbb{F}_p).$$

When $\rho(q) = \text{ord}(\text{Ad}_q)$, we call

$$\mathcal{N}_q = N_q^{(\rho(q))}$$

the adjoint transfer over one adjoint period.

Lemma 2.1 (Matrix order versus adjoint period). *Let*

$$r_0 = \text{ord}(q), \quad \rho = \text{ord}(\text{Ad}_q).$$

Then $\rho \mid r_0$, q^ρ is scalar, and

$$N_q^{(r_0)} = \frac{r_0}{\rho} \mathcal{N}_q \quad \text{in } \text{End}_{\mathbb{F}_p}(\mathfrak{sl}_2(\mathbb{F}_p)). \tag{2}$$

More generally, for every multiple $r = s\rho$,

$$N_q^{(r)} = s \mathcal{N}_q.$$

Proof. The equality $\text{Ad}_q^\rho = 1$ means that q^ρ commutes with every matrix, hence q^ρ is scalar. Since $q^{r_0} = I$, the adjoint period divides r_0 . The length- r_0 sum decomposes into r_0/ρ consecutive blocks of length ρ , and every block is the same because $\text{Ad}_q^\rho = 1$. This gives (2). $\square$

Corollary 2.2 (No period-collapse in the characteristic-three theorem). *If $p = 3$, $q \in GL_2(\mathbb{F}_3)$ is non-scalar, and $r_0 = \text{ord}(q)$, then*

$$\frac{r_0}{\rho(q)} \in \{1, 2\}.$$

Consequently the full matrix-order norm $N_q^{(r_0)}$ has the same rank as the one-adjoint-period transfer $\mathcal{N}_q$. In particular,

$$\text{rank } N_q^{(r_0)} = 1.$$

Proof. For a non-scalar 2×2 matrix over $\mathbb{F}_3$, the scalar subgroup has order 2. Since q^ρ is scalar, the quotient r_0/ρ is the order of this scalar remainder and therefore divides 2. It is thus nonzero in $\mathbb{F}_3$. Equation (2) proves the claim. $\square$

Remark 2.3. *For general p , the distinction between r_0 and ρ must not be suppressed: if $p \mid r_0/\rho$, a nonzero adjoint-period transfer can disappear after repetition over the full matrix order. The characteristic-three theorem is protected from this collapse by the preceding corollary.*

3 Determinant-compatible tangent lifts

Lemma 3.1 (Tangent torsor). *Fix a determinant-compatible reference lift C_0 from modulus p^k to p^{k+1} . Every determinant-compatible lift over the same parent can be written uniquely in the form*

$$C = C_0(I + p^k X) \pmod{p^{k+1}}, \quad X \in \mathfrak{sl}_2(\mathbb{F}_p).$$

Hence there are p^3 such tangent lifts.

Proof. Modulo p^{k+1} , a relative correction is $I + p^k X$. The first-order determinant formula gives

$$\det(I + p^k X) \equiv 1 + p^k \operatorname{tr} X \pmod{p^{k+1}}.$$

Determinant compatibility is therefore equivalent to $\operatorname{tr} X = 0$ in $\mathbb{F}_p$. Since $\dim_{\mathbb{F}_p} \mathfrak{sl}_2 = 3$, there are p^3 possibilities. $\square$

4 Order defect and its first variation

Let $m = p^k$, let C be a lift, and suppose the parent order is r . Whenever

$$C^r \equiv I \pmod{m},$$

define the order defect

$$D_r(C) = \frac{C^r - I}{m} \pmod{p}.$$

Proposition 4.1 (Exact one-step order-defect dichotomy). *Let $m = p^k$, let the reduction of C modulo m have exact order r , and suppose*

$$C^r \equiv I + mD \pmod{pm}.$$

Then the order of C modulo pm is either r or pr , and

$$D = 0 \iff \operatorname{ord}_{pm}(C) = r,$$

while

$$D \neq 0 \iff \operatorname{ord}_{pm}(C) = pr.$$

Proof. The order modulo pm is a multiple of the exact order r modulo m . On the other hand,

$$(I + mD)^p \equiv I \pmod{pm},$$

because every nonconstant term is divisible by $pm = p^{k+1}$. Hence $C^{pr} \equiv I \pmod{pm}$, so the only possibilities are r and pr . The first occurs exactly when $C^r \equiv I \pmod{pm}$, equivalently when $D = 0$. $\square$

Proposition 4.2 (Affine first variation). *Let*

$$C = C_0(I + mX) \pmod{pm}, \quad X \in \mathfrak{sl}_2(\mathbb{F}_p),$$

and let $q = C_0 \pmod{p}$. Then

$$D_r(C) - D_r(C_0) = N_q^{(r)}(X). \quad (3)$$

Thus the defect family is the affine map

$$X \mapsto D_0 + N_q^{(r)}(X), \quad D_0 = D_r(C_0).$$

Proof. Expand the product

$$[C_0(I + mX)]^r$$

to first order in m . Terms containing two factors mX vanish modulo pm . Moving the single tangent insertion through successive powers of C_0 yields the conjugation sum in (3). $\square$

5 The centralizer axis

Lemma 5.1. *For every non-scalar $q \in GL_2(\mathbb{F}_p)$,*

$$\text{Cent}(q) \cap \mathfrak{sl}_2(\mathbb{F}_p) = \langle A_q \rangle.$$

Proof. The centralizer of a non-scalar 2×2 matrix is the two-dimensional algebra $\mathbb{F}_p[I, q]$. Its traceless subspace is one-dimensional, and A_q is a nonzero traceless element of it. $\square$

Proposition 5.2 (Image-axis principle). *For every complete adjoint period,*

$$\text{Im } \mathcal{N}_q \subseteq \langle A_q \rangle.$$

Consequently

$$\text{rank } \mathcal{N}_q \leq 1.$$

Proof. Cyclically shifting the transfer sum shows

$$q^{-1} \mathcal{N}_q(X) q = \mathcal{N}_q(X).$$

The transfer also preserves tracelessness. Its image therefore lies in $\text{Cent}(q) \cap \mathfrak{sl}_2(\mathbb{F}_p) = \langle A_q \rangle$. $\square$

6 Transfer classification

6.1 Regular semisimple case

Proposition 6.1. *Assume $\Delta(q) \neq 0$. Then the adjoint action is semisimple after a field extension, with a one-dimensional fixed space $\langle A_q \rangle$. Over one adjoint period,*

$$\text{rank } \mathcal{N}_q = 1.$$

Proof. After passing, if necessary, to $\mathbb{F}_{p^2}$, the adjoint action has eigenvalues

$$1, \lambda, \lambda^{-1}, \quad \lambda \neq 1.$$

The transfer over the order ρ of λ annihilates the two nontrivial eigenspaces because

$$1 + \lambda + \cdots + \lambda^{\rho-1} = 0.$$

On the fixed line it acts as multiplication by ρ . The semisimple adjoint order is prime to p , hence $\rho \neq 0$ in $\mathbb{F}_p$. Thus the fixed line survives and the transfer has rank one. $\square$

6.2 Modular cyclic norm viewpoint

Before treating the Jordan case explicitly, it is useful to record the standard modular representation-theoretic interpretation. Let $C_p = \langle g \rangle$ act on a vector space over a field of characteristic p . Its cyclic norm element is

$$\mathbf{N} = 1 + g + \cdots + g^{p-1}.$$

Since

$$g^p - 1 = (g - 1)^p$$

in characteristic p , one has the group-algebra identity

$$\boxed{\mathbf{N} = (g - 1)^{p-1}}. \quad (4)$$

The classification of indecomposable modules for cyclic p -groups in characteristic p is classical; the generator acts through unipotent Jordan blocks, with one indecomposable type for each admissible block size $[1, 2]$.

For the adjoint Jordan sector considered below, $g = \text{Ad}_q$ acts on the three-dimensional space $\mathfrak{sl}_2(\mathbb{F}_p)$ and

$$(g - 1)^3 = 0.$$

Consequently (4) immediately implies

$$\mathbf{N} = 0 \quad (p > 3).$$

At $p = 3$, however,

$$\mathbf{N} = (g - 1)^2,$$

which can be nonzero. Thus the exceptional characteristic-three behavior of the Jordan transfer is a direct instance of the standard cyclic modular norm mechanism. The explicit moment calculation below is retained because it identifies the surviving rank-one map concretely.

6.3 Jordan case

Assume now $\Delta(q) = 0$. Since q is non-scalar and invertible, we can write

$$q = a(I + B), \quad B \neq 0, \quad B^2 = 0.$$

Scalar multiplication does not affect conjugation. For $j \in \mathbb{F}_p$,

$$(I + B)^j = I + jB, \quad (I + B)^{-j} = I - jB.$$

Hence

$$\text{Ad}_{q^{-j}}(X) = X + j(XB - BX) - j^2 BXB. \quad (5)$$

Theorem 6.2 (Jordan transfer). *Let q be non-scalar with $\Delta(q) = 0$. Then*

$$\text{rank } \mathcal{N}_q = \begin{cases} 1, & p = 3, \\ 0, & p > 3. \end{cases}$$

Proof. The adjoint period is p . Conceptually, the vanishing for $p > 3$ already follows from the classical cyclic norm identity

$$\mathcal{N}_q = (\text{Ad}_q - 1)^{p-1}$$

together with $(\text{Ad}_q - 1)^3 = 0$.

For completeness, the same mechanism can be seen elementarily by summing (5) over $j = 0, \dots, p-1$:

$$\mathcal{N}_q(X) = \left(\sum 1\right)X + \left(\sum j\right)(XB - BX) - \left(\sum j^2\right)BXB.$$

In $\mathbb{F}_p$,

$$\sum 1 = 0, \quad \sum j = 0.$$

For $p > 3$, $\sum j^2 = 0$, so $\mathcal{N}_q = 0$.

For $p = 3$,

$$\sum_{j=0}^2 j^2 = 5 \equiv 2 \pmod{3},$$

and therefore

$$\mathcal{N}_q(X) = BXB.$$

For nonzero nilpotent B , the map $X \mapsto BXB$ on $\mathfrak{sl}_2(\mathbb{F}_3)$ is nonzero and its image lies on the one-dimensional centralizer axis. Hence its rank is one. $\square$

Theorem 6.3 (Adjoint Transfer Classification). *For a non-scalar $q \in GL_2(\mathbb{F}_p)$, p odd,*

$$\text{rank } \mathcal{N}_q = \begin{cases} 1, & \Delta(q) \neq 0, \\ 1, & \Delta(q) = 0, \ p = 3, \\ 0, & \Delta(q) = 0, \ p > 3. \end{cases}$$

Remark 6.4 (Why characteristic 3 is exceptional). *In the Jordan case the conjugation orbit is polynomial of degree at most two in the orbit parameter j . The constant and linear moments vanish for every odd prime. The quadratic moment vanishes for $p > 3$, but at $p = 3$ its degree equals $p - 1$, and it survives. Thus characteristic 3 is the critical odd characteristic for this 2×2 adjoint problem.*

7 Affine fiber geometry

Suppose the relevant length- r norm in (3) has rank one and the affine equation

$$D_0 + N_q^{(r)}(X) = 0$$

is solvable. Then every zero fiber is a coset of the kernel.

Since

$$\dim \mathfrak{sl}_2(\mathbb{F}_p) = 3, \quad \text{rank } N = 1,$$

we have

$$\dim \ker N = 2, \quad |\ker N| = p^2.$$

Corollary 7.1 (Rank-one branching). *In a rank-one solvable affine state, exactly p^2 of the p^3 determinant-compatible tangent lifts have zero defect and*

$$p^3 - p^2 = p^2(p - 1)$$

have nonzero defect.

Thus the local decomposition is

$$\boxed{p^3 = p^2 + p^2(p - 1)}.$$

8 Characteristic-three order lifting and PRE persistence

We now specialize to $p = 3$. Let $q \in GL_2(\mathbb{F}_3)$ be non-scalar and write

$$r_0 = \text{ord}(q).$$

By the transfer classification and the no-period-collapse result, the full-order transfer $N_q^{(r_0)}$ has rank one and

$$\text{Im } N_q^{(r_0)} = \langle A_q \rangle.$$

Lemma 8.1 (Defect-axis lemma). *Let $C_k \in G_{3^k}$ have exact order r_0 , and let C_0 be any determinant-compatible lift to modulus 3^{k+1} . Write*

$$C_0^{r_0} = I + 3^k D \pmod{3^{k+1}}, \quad D \in M_2(\mathbb{F}_3).$$

Then

$$[q, D] = 0, \quad \text{tr } D = 0,$$

and hence

$$D \in \text{Cent}(q) \cap \mathfrak{sl}_2(\mathbb{F}_3) = \langle A_q \rangle.$$

Proof. Since C_0 commutes with its power $C_0^{r_0}$,

$$0 = [C_0, C_0^{r_0}] \equiv 3^k [C_0, D] \pmod{3^{k+1}}.$$

Dividing by 3^k and reducing modulo 3 gives $[q, D] = 0$.

Next, determinant compatibility gives $\det C_0 = \varepsilon$ with $\varepsilon \in \{\pm 1\}$ at modulus 3^{k+1} . Because the reduction of C_0 modulo 3^k has order r_0 , its determinant satisfies $\varepsilon^{r_0} = 1$; in particular, if $\varepsilon = -1$, then r_0 is even. Thus

$$1 = \det(C_0^{r_0}) = \det(I + 3^k D) \equiv 1 + 3^k \text{tr } D \pmod{3^{k+1}},$$

so $\text{tr } D = 0$ in $\mathbb{F}_3$. For non-scalar q , its centralizer in $M_2(\mathbb{F}_3)$ is $\langle I, q \rangle$; intersecting with the traceless subspace leaves the one-dimensional axis $\langle A_q \rangle$. $\square$

Theorem 8.2 (PRE-Persistence Theorem). *Let $q \in GL_2(\mathbb{F}_3)$ be non-scalar and let $r_0 = \text{ord}(q)$. Suppose $C_k \in G_{3^k}$ reduces to q modulo 3 and has exact order r_0 . Then among the 27 determinant-compatible lifts of C_k to modulus 3^{k+1} , exactly 9 have exact order r_0 and exactly 18 have exact order $3r_0$. In particular, every PRE state has PRE descendants at the next level, so PRE persistence holds for all $k \geq 1$.*

Proof. Choose any determinant-compatible reference lift C_0 . By the defect-axis lemma,

$$D_{r_0}(C_0) \in \langle A_q \rangle.$$

In characteristic three the full-order transfer has rank one and

$$\text{Im } N_q^{(r_0)} = \langle A_q \rangle.$$

Consequently

$$-D_{r_0}(C_0) \in \text{Im } N_q^{(r_0)},$$

so the affine correction equation

$$D_{r_0}(C_0) + N_q^{(r_0)}(X) = 0$$

is automatically solvable. The determinant-compatible lifts form an affine torsor over the three-dimensional space $\mathfrak{sl}_2(\mathbb{F}_3)$. Since $N_q^{(r_0)}$ has rank one, its kernel has dimension two and contains exactly $3^2 = 9$ elements. Hence the zero-defect affine fiber has exactly 9 elements. By the exact one-step order-defect dichotomy, these 9 lifts have exact order r_0 , while the other 18 have exact order $3r_0$.

Every one of the nine order-preserving children again has exact order r_0 , the same non-scalar reduction q , and determinant ± 1 . Therefore the same argument applies at the next level and, by induction, at every subsequent level. $\square$

Corollary 8.3 (Unconditional PRE step). *For every non-scalar characteristic-three PRE state,*

$$\boxed{\text{PRE} \longrightarrow 9\text{PRE} + 18\text{ACT}.}$$

9 Absorbing activation

Lemma 9.1 (Characteristic-three propagation). *For $k \geq 1$,*

$$(I + 3^k E)^3 \equiv I + 3^{k+1} E \pmod{3^{k+2}}.$$

Proof. Expand the cube. The linear term is $3^{k+1}E$; all higher terms have valuation at least $k + 2$. $\square$

If a current order contains an additional factor 3, the relevant periodic adjoint sum contains three identical transfer blocks:

$$N_q^{(3r)} = 3N_q^{(r)} = 0$$

over $\mathbb{F}_3$. Therefore tangent variation cannot cancel a nonzero leading defect at that step. Together with the propagation lemma this gives the absorbing behavior once activation has occurred.

Proposition 9.2 (Absorbing activated step). *Suppose that at level 3^k the current matrix has exact order $R = 3^a r_0$ with $a \geq 1$, and that for a determinant-compatible reference lift C_0 to level 3^{k+1} one has*

$$C_0^R = I + 3^k D \pmod{3^{k+1}}, \quad D \neq 0.$$

Then every one of the 27 determinant-compatible lifts over the same parent has the same nonzero leading R -defect. Hence every child has exact order $3R$ modulo 3^{k+1} . Moreover the resulting nonzero defect propagates to the next level, so the same conclusion repeats inductively.

Proof. Because $R = 3^a r_0$ with $a \geq 1$, the length- R adjoint sum is

$$N_q^{(R)} = 3^a N_q^{(r_0)} = 0$$

over $\mathbb{F}_3$. The affine first-variation formula therefore shows that all tangent children have the same defect $D \neq 0$. The exact one-step order-defect dichotomy then gives exact order $3R$ for every child. Finally,

$$(I + 3^k D)^3 \equiv I + 3^{k+1} D \pmod{3^{k+2}},$$

so the leading nonzero defect survives after the order is multiplied by 3. Repeating the argument proves absorption at every later level. $\square$

Theorem 9.3 (Unconditional recursive branching law). *For every non-scalar characteristic-three lifting tower in the above setting,*

$$\boxed{\text{PRE} \longrightarrow 9\text{PRE} + 18\text{ACT}, \quad \text{ACT} \longrightarrow 27\text{ACT}.}$$

The first rule follows from the PRE-Persistence Theorem; the second from transfer collapse after activation together with the characteristic-three propagation lemma.

10 Counting consequences

Let P_n and A_n denote the numbers of PRE and ACT descendants after n further lifting levels. Then

$$\begin{pmatrix} P_{n+1} \\ A_{n+1} \end{pmatrix} = \begin{pmatrix} 9 & 0 \\ 18 & 27 \end{pmatrix} \begin{pmatrix} P_n \\ A_n \end{pmatrix}.$$

After normalization by 27,

$$\begin{pmatrix} 1/3 & 0 \\ 2/3 & 1 \end{pmatrix}$$

is the transition matrix for the corresponding proportions.

Starting from a PRE root, the surviving PRE proportion after d levels is

$$3^{-d}.$$

The first-activation depth therefore has distribution

$$\Pr(D = d) = \frac{2}{3^d}, \quad d = 1, 2, \dots,$$

and

$$\mathbb{E}[D] = \frac{3}{2}.$$

11 Computational verification

The proof is algebraic; computation was used as an independent discovery and error-detection layer. A dedicated adversarial audit was then run against the interfaces used in the manuscript.

For $p = 3, 5, 7, 11$, exhaustive tests on the determinant- ± 1 non-scalar slices verified:

1. the traceless centralizer is exactly $\langle A_q \rangle$;
2. $\rho(q) \mid \text{ord}(q)$ and $q^{\rho(q)}$ is scalar;
3. the transfer over one adjoint period has the discriminant rank classification proved above;
- 4.

$$N_q^{(\text{ord}(q))} = \frac{\text{ord}(q)}{\rho(q)} \mathcal{N}_q;$$

5. $\Delta \neq 0$ implies $p \nmid \rho(q)$;
6. $\Delta = 0$ implies $\rho(q) = p$;
7. the Jordan conjugation polynomial and its p -block transfer hold exactly;
8. every solvable rank-one affine zero fiber has exactly p^2 elements.

The tested non-scalar populations were

p	non-scalar states	$\Delta \neq 0$	$\Delta = 0$
3	46	30	16
5	236	140	96
7	670	574	96
11	2638	2398	240

and every classification test passed.

The period-ratio census at $p = 3$ was

$$\text{ord}(q)/\rho(q) \in \{1, 2\},$$

with 20 states of ratio 1 and 26 states of ratio 2. Thus no full-order norm collapses in characteristic 3.

The first-variation identity was also checked directly in the modular rings for $p = 3, 5$ and $k = 1, 2, 3$. All tested cases passed: 1242/1242 per level for $p = 3$ and 10000/10000 per level for the deterministic $p = 5$ test set. Finally,

$$(I + p^k E)^p \equiv I + p^{k+1} E \pmod{p^{k+2}}$$

was independently checked for $p = 3, 5, 7$ and $k = 1, \dots, 4$.

These computations are supplementary verification, not logical premises of the theorems.

12 Discussion and scope

The argument combines several classical ingredients: the traceless congruence tangent space, cyclic norm/transfer operators, centralizers of 2×2 matrices, Jordan decomposition, and prime-power binomial congruences. The structural point developed here is that the *order defect itself varies affinely by the adjoint transfer*. This turns representation-theoretic rank information into exact counts of order-preserving and order-increasing lifts.

Characteristic 3 is special because the non-semisimple adjoint orbit has quadratic dependence on the cyclic parameter and its quadratic moment survives precisely when $p - 1 = 2$. This removes the Jordan rank-zero sector present for larger odd primes and makes the rank-one branching law uniform over all non-scalar coarse states.

For $p > 3$, the zero-discriminant sector requires a different higher-order analysis: the first-order transfer vanishes, so a three-way description by semisimple rank-one states, dormant Jordan states, and activated states may be more natural than the characteristic-three two-state grammar.

A dedicated proof-interface audit additionally verified all finite characteristic-three identities used in the inductive proof: the traceless-centralizer axis identity for all 46 non-scalar coarse states; equality $\text{Im } N_q^{(r_0)} = \langle A_q \rangle$; centrality and tracelessness of all tested reference defects; exact nine-element affine correction fibers; preservation of exact order r_0 along PRE descendants; the 9/18 split through the tested levels; activated norm collapse; and the cubic propagation identity. These computations are not used as a replacement for the proof above, but as an independent adversarial audit of each algebraic interface.

13 Literature position after reference-chain audit

The present argument deliberately separates classical ingredients from the specific order-defect construction developed here.

13.1 Congruence tangent spaces

The appearance of traceless matrices in the first congruence layer is classical. Lee and Szczarba studied principal congruence subgroups and their associated traceless matrix modules [3]. Lopez later described the Lie algebra associated with the p -congruence filtration of $SL_n(R)$, giving the expected $\mathfrak{sl}_n$ -valued graded structure [4]. Accordingly, the identification of the determinant-compatible tangent space with $\mathfrak{sl}_2(\mathbb{F}_p)$ is background rather than a novelty claim.

13.2 Cyclic modular transfer

The modular representation theory of cyclic p -groups is likewise classical. Indecomposable modules are described by unipotent Jordan blocks; standard treatments include Alperin [1] and Benson [2]. In characteristic p , the cyclic norm element satisfies

$$1 + g + \cdots + g^{p-1} = (g - 1)^{p-1}.$$

Thus, once the Jordan adjoint action is known to satisfy $(g - 1)^3 = 0$, the vanishing of its norm for $p > 3$ is a standard modular consequence. The characteristic-three survival of the quadratic Jordan contribution is therefore not claimed as a new representation-theoretic phenomenon in isolation.

13.3 Linear recurrences and Fibonacci periods

The arithmetic of periods of linear recurrences over finite fields and modulo integers has a long history. Wall's classical study of the Fibonacci sequence modulo m established a foundational framework for Fibonacci periods [6]. Matrix formulations of generalized Fibonacci recurrences also appear in the classical literature [7]. Consequently, Fibonacci/Pisano period lifting and the use of companion matrices are treated here as background motivation, not as the novelty of the present theorem.

13.4 Prime-power order lifting

The general principle that the order of an element may acquire a p -power factor when lifted through successive congruence levels is part of the standard structure of congruence groups and p -adic lifting. Likewise, the binomial congruence

$$(I + p^k E)^p \equiv I + p^{k+1} E \pmod{p^{k+2}}$$

is elementary and is not claimed as new.

13.5 Contribution of the present paper

The paper does not claim novelty for congruence tangent spaces, cyclic norm/transfer operators, Jordan modules, Fibonacci period theory, or prime-power order growth separately.

The structural contribution of the present paper is the affine identification

$$\boxed{D_r(C_0(I + p^k X)) = D_r(C_0) + N_q^{(r)}(X),}$$

where

$$D_r(C) = \frac{C^r - I}{p^k} \pmod{p},$$

followed by the use of the rank and affine fibers of the adjoint transfer to enumerate *exactly* the determinant-compatible lifts that preserve or increase matrix order. In the characteristic-three 2×2 setting this produces the uniform law

$$\text{PRE} \longrightarrow 9\text{PRE} + 18\text{ACT}, \quad \text{ACT} \longrightarrow 27\text{ACT}.$$

Equivalently, the contribution may be summarized by the chain

$$\begin{aligned} \text{matrix order lifting} &\longrightarrow \text{affine order defect} \longrightarrow \text{adjoint transfer} \\ &\longrightarrow \ker N \longrightarrow \text{exact lifting multiplicity} \longrightarrow \text{absorbing activation.} \end{aligned}$$

These ingredients have substantial prior literatures. In particular, recent work on Fibonacci residues modulo prime powers uses genuinely p -adic methods and Wall-type exponents [5]. The result proved here is therefore formulated at the level of the specific synthesis: the affine first-variation formula for the order defect, its adjoint-transfer linear part, the exact counting of determinant-compatible order-preserving fibers, and the resulting characteristic-three branching law.

A Finite verification checkpoints

As an independent check on the algebraic proofs, the following identities were verified exhaustively over the relevant finite rings and fields:

1. determinant-compatible lifts form an $\mathfrak{sl}_2(\mathbb{F}_3)$ -torsor;
2. the order defect has the preserve-or-triple dichotomy;
3. the exact first variation is affine with linear part $N_q^{(r)}$;
4. the defect lies on the traceless centralizer axis;
5. the characteristic-three transfer has rank one for all non-scalar coarse states;
6. the defect of every PRE reference lift is traceless and centralizes q ;
7. the defect axis equals the image of the full-order transfer;
8. the affine correction equation is therefore automatically solvable and has exactly 9 solutions;
9. PRE descendants retain exact order r_0 , closing the induction;
10. the activated periodic norm vanishes;
11. the cube propagation preserves a nonzero leading defect.

B References and background

The references below document classical ingredients and neighboring results used to position the main theorem.

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